On the Arrangement of the Real Branches of Plane Algebraic Curves.

By V. Ragsdale.

Introduction.

In the consideration of any problem relating to the number and arrangement of the real branches of plane algebraic curves, the division of circuits into the two classes *odd* and *even* is of fundamental importance.* An odd circuit can be met by a straight line in an odd number of points only; an even circuit in an even number of points. Two odd circuits have an odd number of intersections; an even and an odd circuit, or two even circuits have an even number of points in common. Hence, as Zeuthen shows (*Sur les differentes formes des courbes planes du quatrième ordre,* Math. Ann. VII, 1873, pp. 410-432), a non-singular curve of even order must be composed entirely of even circuits, and a non-singular curve of odd order must have one circuit odd and the rest even.†

In a paper published in 1876 (*Ueber die Vieltheiligkeit der ebenen algebraischen Curven,* Math. Ann. X, pp. 189-198) Harnack proved that a curve cannot have more than $p + 1$ circuits, where p denotes the genus of the curve; also that for every value of p, a curve of some order does exist having $p + 1$ real branches. In particular, if p be of the form $\frac{1}{2}(n - 1)(n - 2)$, there exists a non-singular n^{ic} with $\frac{1}{2}(n - 1)(n - 2) + 1$ real branches. Later, Hilbert (*Ueber*

---

* Von Staudt, *Geometrie der Lage,* 1847, p. 80.

† Zeuthen (*loc. cit.* p. 426) proves the existence of a quartic circuit with two double points, which is met by every straight line in at least two real points, and hence can not be projected into the finite part of the plane. Cayley (*On Quartic Curves,* Collected Papers, V. op. 361, 1865) points out that the sextic may be composed of one non-singular circuit which is met by every straight line in at least two real points. And C. A. Scott (*On the Circuits of Plane Curves,* Transactions of the American Mathematical Society, 1902) establishes the general theorem as to the existence of circuits that cannot be projected into the finite part of the plane. In the following pages, however, the only circuits that present themselves are those which can be projected into the finite, and for these the term *oval* is here employed.

die reellen Züge algebraischer Curven, Math. Ann. XXXVIII, 1890, pp. 115-138) considered certain possibilities of arrangement for the circuits of a non-singular n^{ic} when the maximum number of branches is present, and Hulburt (*A Class of New Theorems on the Number and Arrangement of the Real Branches of Plane Algebraic Curves,* American Journal of Mathematics, XIV, 1892, pp. 246-250) extended Hilbert's theorems to certain cases of curves with double points. Hilbert proved that for n even, not more than $\frac{1}{2}(n-2)$ of the $p+1$ ovals can be nested; that is, so situated that the first lies *inside** a second, the second inside a third, and so on; and that curves of even order do exist having $p+1$ ovals, $\frac{1}{2}(n-2)$ of which are nested; similarly, that for n odd, not more than $\frac{1}{2}(n-3)$ ovals can be nested, if the maximum number of circuits is present, and that curves of odd order do exist having $p+1$ circuits, $\frac{1}{2}(n-3)$ of which are nested ovals.

A footnote to this paper† contains the statement that if the non-singular sextic have its maximum number of branches, eleven, these cannot all lie external to one another. Hilbert speaks of the process by which he arrived at this conclusion as "ausserordentlich umständlich," but no hint as to the character of the argument is given, and no proof of the statement has ever been published. However, if such a limitation on the arrangement of the ovals does exist for the 6^{ic}, there arises at once the question as to the existence of a similar limitation for all non-singular curves with the maximum number of branches. For curves of odd order no such restriction holds,—at least, in the form stated by Hilbert,—for it can be shown that a non-singular curve of odd order may have the maximum number of circuits with every oval lying outside the others. For the discussion of the question for curves of even order, however, it is convenient to cast Hilbert's statement into a slightly different form.

For the two types of the 6^{ic} given by Hilbert, the only types that can be derived by his method of generation, the arrangement of the ovals is the following:

(1) An oval O; 1 oval inside O, and 9 outside.

(2) An oval O; 9 ovals inside O, and 1 outside.

It is seen that the numbers of ovals "inside" and "outside" are interchanged in the two cases, and the natural inference is, that the law of arrangement to

* For the definition of "inside" and "outside" of a closed circuit see Von Staudt (*l. c.* p. 90) and Zeuthen (*l. c.* p. 410).

† l. c. pp. 118–119.

which the ovals are subject, is independent of the distinction between the "inside" and "outside" of a closed circuit, as defined by Von Staudt and Zeuthen, and, in fact, is based on no distinctive or permanent property of any one region of the plane. Thus the division of the plane by the curve $u = 0$ into regions where u is positive and regions where u is negative offers a more promising basis for investigation of the problem, because of the element of arbitrariness introduced in ascribing to a certain region the positive rather than the negative sign. Suppose that the curve is non-singular and of even order, and that all its ovals have been projected into the finite. According to the usual convention let the sign be determined so that the expression u is positive at infinity. A region where u is negative may be a region bounded by a single circuit as in Fig. 1, or a region bounded by two or more circuits as in Fig. 2.

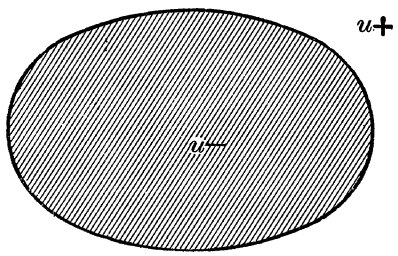


FIG. 1.

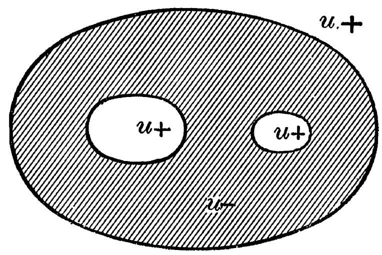


FIG. 2.

Each additional boundary introduces a new positive region. If such a boundary, or an oval which cuts off in the midst of a region where u is negative a region in which u is positive, be called an *internal* oval, and an oval which cuts off in the midst of a region where u is positive a region in which u is negative, an *external* oval, Hilbert's statement can be expressed as follows: *If the non-singular sextic have its maximum number of branches, at least one of the eleven ovals must be internal*;—that is, not more than ten of the eleven ovals can be external. There is as yet no formal proof forthcoming for this statement in either its original or altered form, but as curves of higher order are investigated a most interesting law governing the arrangement of the ovals presents itself so persistently, and in curves of such widely different types, as to give strong reasons for belief in the existence of a general theorem. It is found that the 8^{ic}, 10^{ic}, 12^{ic}, 14^{ic}, ..., with the maximum number of circuits, will have respectively 3, 6, 10, 15, ..., or more internal ovals. And in general, *if the non-singular $2n^{ic}$ have the maximum number of branches, at least $\frac{1}{2}(n-1)(n-2)$ of the $p+1$ ovals must be internal; or not more than $n^2 + \frac{1}{2}(n-1)(n-2)$ can be external.*

49

As will be shown later, the only processes by which curves with the maximum number of branches have been derived, yield curves of even order whose circuits conform to this law of arrangement. These are the two processes employed by Harnack and Hilbert. The *Harnack process* offers two modes of generation, each of which determines a distinct law of arrangement for the circuits of the derived curve C_{2n}; but these two laws and all modifications of them which arise from combinations of the two modes of generation differ only in the distribution of the internal ovals. The number in every case is $\frac{1}{2}(n-1)(n-2)$. For example, of the 22 ovals of the 8^{ic}, 3 are internal, though these 3 may be distributed in two ways (Fig. $3_{(b),(c)}$). Of the 37 ovals of the 10^{ic}, 6 are internal, though these 6 may be distributed in four ways (Fig. $3_{(d),(e),(f),(g)}$).

The *Hilbert process* gives less simple arrangements of the circuits. Hilbert's own statement is, that if the $2n^{ic}$ have the maximum number of nested ovals, $n-1$, the remaining ovals must be external to one another and may be distributed in various ways in the annular regions bounded by two successive nested ovals, and in the region lying outside the nest. It is shown (p. 389) that the simplest arrangement of these remaining ovals is represented by the following scheme, which gives the number of ovals in the annular regions, beginning with the innermost ring, 0, 2, 4, 6, 8, ... $2n-10$, $2n-8$, $2n-6$; the other ovals lie outside the nest entirely. In this case the number of internal ovals is exactly $\frac{1}{2}(n-1)(n-2)$. But for all curves of order $2n\,(2n>6)$, the process gives choice of three distinct modes of generation, and hence affords various possibilities for the arrangement of the circuits. It is still true, however, that no type of $2n^{ic}$ obtained has less than $\frac{1}{2}(n-1)(n-2)$ internal ovals.

Both these processes are based upon the principle of small variation from a special degenerate curve. This reducible curve is composed of an $m-k^{ic}$ with the maximum number of circuits and an auxiliary curve of order k which bears a certain specified relation to the $m-k^{ic}$. The m^{ic} obtained has the maximum number of circuits, and bears a relation to the auxiliary curve similar to that possessed by the $m-k^{ic}$. The two methods differ only in the type of auxiliary curve employed. The Harnack process is characterized by the use of the straight line as auxiliary curve; the Hilbert process by the use of the ellipse. Hulburt has proved* that in the generation of curves by the method of small variation, the only auxiliary curves that will yield the maximum number of

*l. c. p. 250.

branches on the derived m^{ic}, are the straight line and the conic; that is, the only processes of this type that will give curves with the maximum number of circuits are the Harnack and Hilbert methods, and modifications of the Hilbert method due to the use of other conics as the auxilary curve. Hence if all non-singular curves with the maximum number of branches are obtainable by the method of small variation, the law which expresses the arrangement of ovals for curves derived by the Harnack and Hilbert processes becomes a general law, and holds for *all* curves of even order with the maximum number of circuits. But whether the law is of perfect generality or not, it is of interest to investigate more fully the various types of curves that can be derived by these different methods.

Curves with the Maximum Number of Branches Derived by the Harnack and Hilbert Processes of Generation.

Curves Derived by the Harnack Process.—Let C_{n-1} be a non-singular curve of order $n-1$ with the maximum number of circuits, and let a straight line v meet one circuit of the curve in $n-1$ real and distinct points which have the same order of succession on C_{n-1} as on v. Harnack shows then that by a proper choice of δ and the straight lines, $l_1, l_2, l_3, \ldots l_n$, $C_n \equiv v \,.\, C_{n-1} + \delta \,.\, \prod_{i=1}^{i=n} l_i = 0$ can be made to represent a curve of order n having properties corresponding to those of the $n-1^{ic}$. Certainly for $n=3$, such an $n-1^{ic}$ exists; viz., a conic cut in two real points by a straight line v. Let three lines l_1, l_2, l_3 be chosen so that they cut the infinite segment of v. Then the cubic represented by the equation $C_2 \,.\, v + \delta \,.\, l_1 l_2 l_3 = 0$ will pass through the intersections of C_2 and v with $l_1 l_2 l_3$, and for a small value of δ will have the maximum number of circuits, two. Moreover, the infinite branch is cut by the straight line v in three real points (Fig. 1, Plate I). The quartic can be derived from the cubic in the same way. Let the lines l_1, l_2, l_3, l_4 be so chosen that they cut the same segment of v; then for a proper choice of δ, the equation, $C_4 \equiv C_3 \,.\, v + \delta \,.\, \prod_{i=1}^{i=4} l_i = 0$, will represent a quartic with four ovals, one of which meets v in four real points (Fig. 2, Plate I). Similarly, the quintic with the maximum number of circuits can be obtained from the quartic, and the sextic from the quintic, and so on.

The restrictions imposed on the $n-1^{ic}$, viz.: (1) That the $n-1^{ic}$ must have the maximum number of branches, (2) that a straight line v must cut C_{n-},

in $n-1$ real and distinct points, (3) that all these points must lie on the same circuit of C_{n-1}, are necessary in order that C_n may have the maximum number of branches. Let that circuit of the $n-1^{ic}$ which is cut by v in $n-1$ real points be called the generating circuit g_{n-1}. Restrictions (2) and (3) require that g_{n-1} be the infinite branch, if $n-1$ is odd. There are also certain restrictions that must be imposed on the lines $l_1, l_2, l_3, \ldots . l_n$. The points in which C_n cuts v are determined by the points common to v and the lines $l_1, l_2, l_3, \ldots . l_n$. If an odd number of these lines cut any finite segment determined on v by C_{n-1} (*n odd*) or *any* segment (*n even*) the number of circuits of C_n falls short of the maximum number. It is clear that by admitting imaginary lines there can be obtained from the given $n-1^{ic}$, n^{ics} with the maximum number of branches, and cut by v in $0, 2, 4, \ldots .$ or n real points if n is even, or in $1, 3, 5, \ldots . n$ real points if n is odd; and also that these points of intersection, if more than two, may lie on different circuits. But for the generation of the $n+1^{ic}$ with the maximum number of circuits from the n^{ic} all the intersections of C_n and v must be real and lie on the same circuit g_n. Hence the straight lines $l_1, l_2, l_3, \ldots . l_n$ must be chosen to cut the same segment of v, and in n real and distinct points. If n is odd, this segment must be the infinite segment; if n is even, the segment may be any one, finite or infinite. The general arrangement of the circuits of the n^{ic} with the maximum number of branches is the same whether the intersections of C_n and v are all real and lie on the same circuit or not. The difference in the two cases manifests itself in the number of branches on the $n+1^{ic}$ and curves of higher order derived from the n^{ic}. Hence, in considering the different types of n^{ics} with the maximum number of branches, it is necessary to take account only of those cases where the lines $l_1, l_2, \ldots . l_n$ are subject to such restrictions as allow the process to be continued.

In the generation of the quintic from the quartic (Fig. 3, Plate 1), one circuit of C_4 must cut v in four points. Of the four segments of g_4, two with their corresponding segments of v give rise to two ovals lying external to one another. Of the other two, one, together with the infinite segment of v, generates the infinite branch of the 5^{ic}; the other, with its corresponding segment of v, produces an oval on the 5^{ic}, which must lie in one of the regions bounded by a segment of v and a segment of the infinite branch of C_5 (Fig. 3, Plate I). Hence the 6^{ic} arising from this 5^{ic} must have one oval lying inside another; the other ovals are external, five representing the five remaining ovals of the 5^{ic}, and four generated by the other segments of v and the infinite branch of C_5 (Fig. 4, Plate I).

Though with reference to the arrangement of the circuits there is only one kind of 6^{ic} obtained, there are with regard to the 6^{ic} two essentially different positions of the lines $l_1, l_2, \ldots l_6$, which are of importance in the generation of curves of higher order. According as these lines cut that segment of v which, together with one segment of C_5, encloses a region containing an oval, or one of the other segments, the generating oval encloses (a) one other oval (Fig. 4, Plate I), or (b) includes no oval (Fig. 5, Plate I).

From the 6^{ic} of type (a) is derived a 7^{ic} with the maximum number of circuits each oval of which lies external to the others ; from the 6^{ic} of type (b), a 7^{ic} with the maximum number of branches and with two ovals nested. It is seen from Figs. 4, 5, Plate I, that of the six segments of g_6, three lie on one side of v and with segments of v give rise to three ovals on C_7 which are external to one another; but of the other three, two lie in the region bounded by the third and the finite segment $x_1\, x_6$ of v. This third segment of g_6 and the infinite segment x_6x_1 of v give rise to the infinite branch of the 7^{ic}. The other two segments, with their corresponding segments of v, generate two ovals external to one another, but situated in one of the seven regions formed by the intersections of g_7 and v. In this region must lie also the representative of the oval, if any, which is encircled by the generating oval. Hence the 7^{ic} of the first type must have three ovals lying in one of the 7 regions bounded by a segment of v and a segment of g_7, and the 7^{ic} of the second type (the one that has the pair of nested ovals) must have two ovals lying in one of these seven regions. In both cases the remaining six regions contain no ovals ; the arrangement of the other ovals is similar to the arrangement of those on the sextic from which they are derived. Hence *the 8^{ic} generated by the 7^{ic} of the first type will have one oval which encloses three others.* There is only one type of 8^{ic} obtained, but the generating oval may be that which encircles three others, or one which includes none. Thus, in passing to the 9^{ic}, there is a choice again between two modes of generation. *The 8^{ic} generated by the 7^{ic} of the second type has one oval enclosing two others and one including a single oval;* and two cases arise as before, according as the generating oval is the oval which encloses two others, or one which includes none. Hence, just as two types of the 7^{ic} were derived from the one form of the 6^{ic}, two types of the 9^{ic} are generated by each of the two forms of the 8^{ic}.

By exactly the same argument it can be shown that each type of the $2m-2^{ic}$ gives rise to two types of the $2m-1^{ic}$, determined by the character of the generating oval, g_{2m-2}, which may enclose a number of other ovals or may contain none at all. But since in the generation of the $2m^{ic}$ from the $2m-1^{ic}$

the straight lines $l_1, l_2, \ldots . l_{2m-1}$ must cut the infinite segment of v, each type of the $2m-1^{ic}$ can give rise to only one type of the $2m^{ic}$. In this way two types of the $2m^{ic}$ arise from each form of the $2m-2^{ic}$ ($2m > 6$), and as the process is continued ($2m = 8, 10, \ldots . 2n$) there will arise 2^{n-3} types of the $2n^{ic}$. These, however, do not differ in the number of internal ovals. For with the exception of the ovals derived from v and the infinite branch of C_{2n-1}, the arrangement of the ovals of the $2n^{ic}$ is similar to that of the $2n-1^{ic}$. The infinite branch of C_{2n-1} and v form $2n-1$ regions, each bounded by a single segment of C_{2n-1} and a single segment of v, and in one of these regions must lie the representatives of all the ovals, if any, included by the generating oval of C_{2n-2}, as well as $\frac{1}{2}(2n-2)-1$ of the $2n-3$ ovals which arise from segments of g_{2n-2} and v. No other of the $2n-1$ regions contains an oval. Hence the oval on the $2n^{ic}$ derived from the segments of C_{2n-1} and of v which bound this region, will include $n-2$ ovals and also those representing the ovals which were contained by g_{2n-2}. Thus whatever be the type of $2n^{ic}$, the number of its internal ovals exceeds by $n-2$, the number on the $2n-2^{ic}$.

For the 6^{ic} the number of internal ovals is 1,
for the 8^{ic}, $1 + 2$,
for the 10^{ic}, $1 + 2 + 3$,
for the 12^{ic}, $1 + 2 + 3 + 4$, etc.

Hence on *every* curve of even order ($2n$) with the maximum number of circuits there are $1 + 2 + 3 + 4 + \ldots . + n - 4 + n - 3 + n - 2$, i.e. $\frac{1}{2}(n-1)(n-2)$ internal ovals, and hence $n^2 + \frac{1}{2}(n-1)(n-2)$ external ovals.

Though the number of internal ovals is the same for all $2n^{ics}$ thus generated, the distribution differs from type to type. If on each $2m^{ic}$ ($2m = 2, 4, 6, \ldots . 2n-2$) from which the $2n^{ic}$ is derived the generating oval be one which contains others, that is, if the first mode of generation be used throughout, all the internal ovals lie inside the same oval (Fig. 3 $_{(b), (d), (h)}$). If, however, the generating oval be one which encloses no other, then among the *additional* circuits formed in passing from a curve of order $2n-2$ to a curve of order $2n$, there is just one oval which includes others, and this contains $n-2$. Thus the $2n^{ic}$ which is derived by the second mode of generation throughout has its internal ovals distributed in $n-2$ different ovals, in groups of $1, 2, 3, 4, \ldots . n-3, n-2$ (Fig. 3 $_{(c), (g), (o)}$). Combinations of the two modes of generation afford ($2^{n-3}-2$) other arrangements of the internal ovals, but there are certain restrictions to which the distribution is subject. No set of $1, 2, 3, \ldots, n-2$ ovals can be separated, and combinations can be made only of successive sets (Fig. 3 $_{(e), (f), (i), (j), (k), (l), (m), (n)}$).

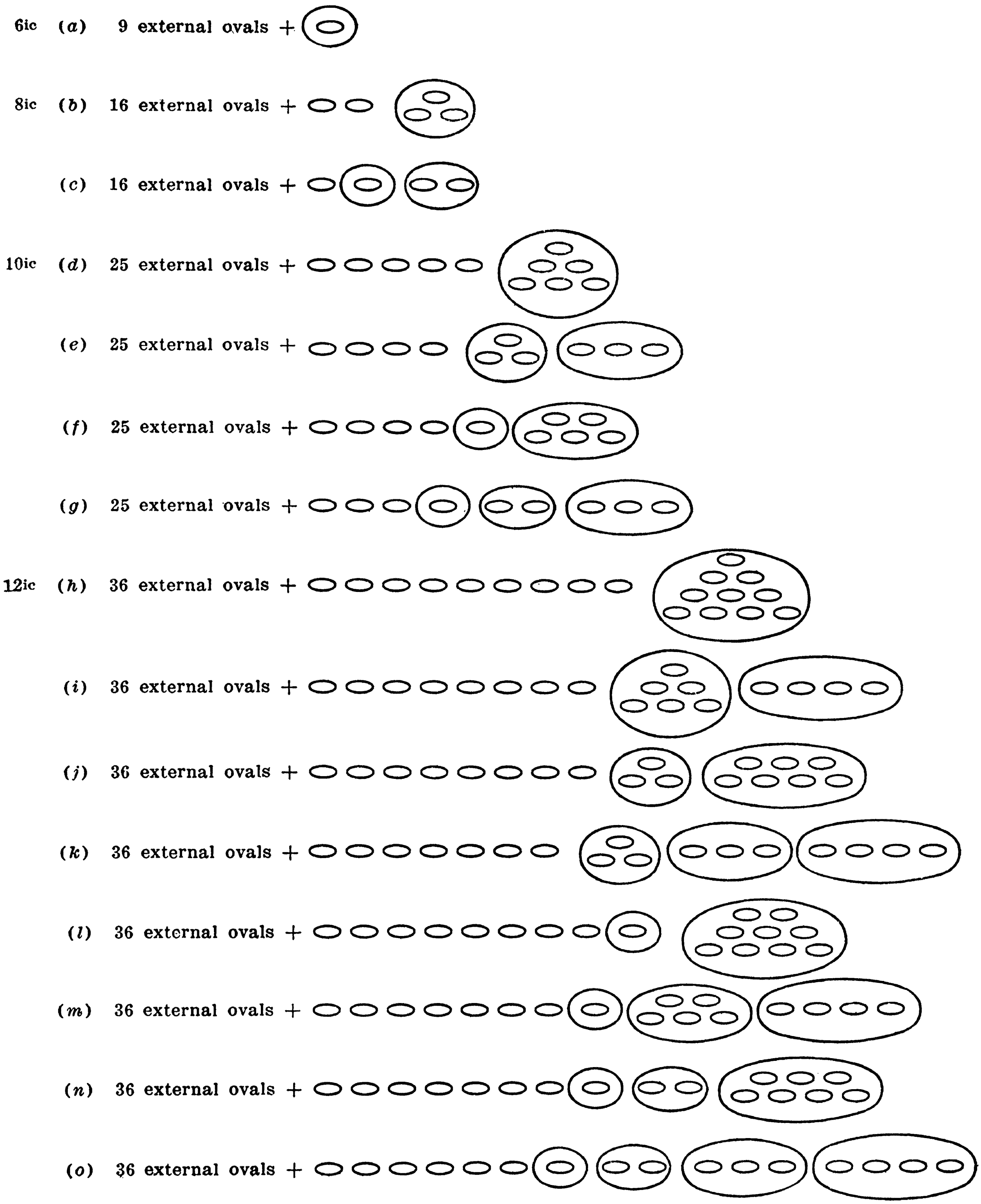

6ic (a) 9 external ovals +
8ic (b) 16 external ovals +
(c) 16 external ovals +
10ic (d) 25 external ovals +
(e) 25 external ovals +
(f) 25 external ovals +
(g) 25 external ovals +
12ic (h) 36 external ovals +
(i) 36 external ovals +
(j) 36 external ovals +
(k) 36 external ovals +
(l) 36 external ovals +
(m) 36 external ovals +
(n) 36 external ovals +
(o) 36 external ovals +

FIG. 3.

If the non-singular curve of even order has not the maximum number of circuits, the question arises, whether in this case the number of external ovals can exceed $n^2 + \frac{1}{2}(n-1)(n-2)$. The generating oval of C_{2n-2} with v gives rise to the infinite branch of C_{2n-1} and to $2n-3$ ovals, $n-2$ of which become internal ovals on the $2n^{ic}$. The remaining $n-1$ ovals, and all additional ovals arising from g_{2n-1} and v become external ovals. It is obvious that the $2n-2$ straight lines $l_1, l_2, l_3, \ldots l_{2n-2}$ could have been so chosen that the position of g_{2n-2} with regard to v would have been that indicated by Fig. 4, and hence that

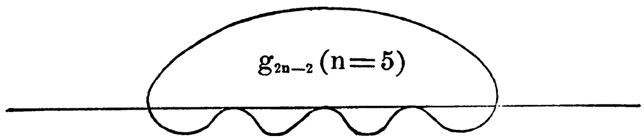


FIG. 4.

none of the $n-2$ internal ovals would have appeared on the $2n^{ic}$. This is the only way, however, in which the presence of internal ovals can be prevented without decreasing the number of external ovals, and this method admits of no increase in the number of the latter. Hence the conclusion can be drawn that no 6^{ic} derived by the Harnack process, can have more than 10 external ovals, no 8^{ic} more than 19 external ovals, no 10^{ic} more than $31, \ldots$, no $2n^{ic}$ more than $n^2 + \frac{1}{2}(n-1)(n-2)$, even though the number of circuits on the curve be less than the maximum number.

In support of the statement that on *curves of odd order*, the ovals may be so arranged that each lies outside the others, it was seen (p. 383) that from the 6^{ic} whose generating circuit included another oval, a 7^{ic} can be derived whose ovals lie external to one another. And in general from every $2n^{ic}$ whose generating oval includes *all* the internal ovals can be derived a $2n+1^{ic}$, all of whose ovals are external to one another. For as C_{2n+1} is generated from C_{2n}, the generating oval opens out, so to speak, to form with the infinite segment of v, the infinite branch of the $2n+1^{ic}$, thus leaving the ovals which it contained free of any encircling oval.

Curves of Even Order derived by the Hilbert Process.—For the types of $2n^{ics}$ which present themselves by the Harnack process, all the ovals which lie "inside" others satisfy the definition of "internal" ovals. Not so for curves with nested ovals; for within the annular regions bounded by two successive ovals of the nest, beginning with the outermost ring the expression, C_{2n}, is alter-

nately negative and positive. The internal ovals, then, are to be looked for only in these negative regions, that is, in the 1st, 3rd, 5th, ... etc., from the outside. The circuits lying in the 2nd, 4th, 6th, ... regions lie "inside" certain ovals but are themselves external ovals.

It has already been mentioned that the Hilbert method of generating curves with the maximum number of circuits, differs from the Harnack process only in the use of the ellipse instead of the straight line as auxiliary curve. The process, as given by Hilbert, applies to curves of both odd and even orders, but here only curves of even order will be considered. Let C_{2n} be a curve of even order with the maximum number of circuits, $p+1$, and the maximum number of nested ovals, $n-1$; and assume that an ellipse, E_2, can be drawn to enclose one or more of the nested ovals and cut one of the non-nested ovals, g_{2n}, in $4n$ points which have the same order of succession upon C_{2n} as upon the ellipse. On a segment, S, of E_2, but not that which with a segment of g_{2n} encloses the one or more nested ovals inside the ellipse, let $4n+4$ points be chosen and through these points let $2n+2$ straight lines, $l_1, l_2, \ldots l_{2n+2}$, be drawn, connecting the first point with the second, the third with the fourth, and so on.* Then for a small value and the proper sign of δ the equation $C_{2n}\,.\,E_2+\delta\,.\prod_{i=1}^{i=2n+2} l_i=0$ represents a curve of order $2n+2$, which has the maximum number of branches, $p+1$, the maximum number of nested ovals, n, and satisfies all other conditions analogous to those assumed for the $2n^{ic}$.† Hence if a $2n^{ic}$ exists satisfying the assumed conditions, from it can be derived a $2n+2^{ic}$, subject to similar conditions, from this a $2n+4^{ic}$, and so on. For the case $2n=4$, such a curve does exist.

---

* It can be seen, as in the preceding method, that the assumptions made for the $2n^{ic}$ are necessary for the maximum number of circuits, or for the maximum number of nested ovals on the $2n+2^{ic}$, or for the continuation of the process beyond the generation of $C_{\overline{2n+2}}$. The assumption that the ellipse must enclose at least one of the nested ovals and the restriction made on the segment, S, are not given by Hilbert but are shown by Hulburt (*Topology of Algebraic Curves*, Bull. N. Y. Math. Soc. I, 1891-2, p. 197) to be necessary for the continuation of the process. Otherwise the curves of higher order would not have the maximum number of nested ovals. The restriction of the $4n+4$ points to the *same* segment is necessary in order that curves of order $> 2n+2$ may have the maximum number of circuits.

† With the exception of the oval, g_{2n}, each circuit of C_{2n} gives rise to a circuit of C_{2n+2}. Also the boundaries of the $4n$ regions formed by the intersections of g_{2n} and E_2 generate $4n$ ovals. Hence *the total number of branches* $=\frac{1}{2}(2n-1)(2n-2)+4n=p+1$. The ovals arising from the nested ovals are nested, and one of the $4n$ ovals generated by the segments of g_{2n} and E is itself a nested oval. Hence the number of nested ovals $=\frac{1}{2}(2n-2)+1=\frac{1}{2}(\overline{2n+2}-2)$. Moreover the ellipse encloses one or more of the nested ovals, and a non-nested oval, g_{2n+2}, cuts E_2 in $4n+8$ points whose order of succession is the same on the oval and the ellipse.

Let C_2 and E_2 represent two ellipses cutting each other in four real points. On a segment, S, of E_2 let 8 points be chosen and joined by the straight lines l_1, l_2, l_3, l_4, the 1st point from one end of the segment with the 2nd, the 3rd with the 4th, and so on. Then the equation $C_n \,.\, E_2 + \delta \,.\, \prod_{i=1}^{i=4} l_i = 0$, for a proper choice of δ, will represent a quartic satisfying the assumed conditions (Figs. 1, 2, Plate II). And therefore for all order values, $2n$, curves do exist satisfying similar conditions.

There is only one type of quartic obtained, but two cases arise from the two possible positions of the lines l_1, l_2, l_3, l_4 with reference to the auxiliary ellipse. If in the derivation of the 4^{ic} from the conic, C_2, no real points had been chosen on a segment, S, of E_2, the quartic would have consisted of two ovals inside E_2 and two ovals outside. Hence according as the 8 points chosen lie on a segment, S, outside C_2 or on a segment inside C_2, one of the two ovals inside E_2, or one of the two ovals outside E_2, becomes the generating oval. Each of these two quartics gives rise to a distinct type of 6^{ic} $(C_4 \,.\, E_2 + \delta \,.\, \prod_{i=1}^{i=6} l_i = 0)$ with the required properties (Figs. 3, 4, Plate II). It is easily seen that in the generation of each type of 6^{ic}, there are possible three essentially different positions of the 12 points which, taken in pairs, determine the 6 straight lines $l_1, l_2, \ldots . l_6$. For one position, the ellipse, E_2, is cut by a non-nested oval of C_6 which would otherwise lie inside the ellipse; for another, by a non-nested oval which would otherwise lie outside the ellipse; for the third, by a nested oval. And, in general, the same possibilities arise in the derivation of the $2n^{ic}$ from the $2n - 2^{ic}$, thus affording three modes of generation of the $2n + 2^{ic}$ from a given $2n^{ic}$. The third case, however, as Hulburt points out, leads to curves with less than the maximum number of nested ovals. It will be found later that after an application of the third mode of generation, a fourth mode becomes possible. All four modes yield curves with the maximum number of circuits, but only the first and second admit also the maximum number of nested ovals.

If at each stage of the generation of the $2n^{ic}$ from the curves of lower order, one of the non-nested ovals inside the ellipse be taken as the generating oval, that is, if a *curve be derived* by *the first mode of generation* throughout, the circuits situated in the $n - 2$ annular regions determined by the nested ovals are distributed according to a perfectly regular scheme. The non-nested oval of the $2n - 4^{ic}$ which cuts the ellipse forms with the latter, $4n - 8$ regions in which

are generated $4n-8$ new ovals of the $2n-2^{ic}$. Of the $2n-4$ of these which lie inside the ellipse, one is a nested oval; another is to be taken as the generating oval; hence in the region bounded by the outer oval of the nest and those segments of the ellipse and the generating oval which give rise to the *new* nested oval of C_{2n}, there are situated exactly $2n-6$ ovals. Therefore the $2n^{ic}$ will have in the last annular region formed $2n-6$ ovals. Since one new annular region is formed at each stage of the generation, and the arrangement of the ovals lying in this region is not disturbed as curves of higher order are generated, there corresponds to each curve, C_{2m}, ($2m \geqq 6$) from which the $2n^{ic}$ is derived, one particular ring. For the 6^{ic}, the number of ovals in the ring between the two nested ovals $=0$; for the 8^{ic}, the number of ovals in the 1st, or innermost, and the 2nd rings $=0$, 2; for the 10^{ic} the number of ovals in the 1st, 2nd, and 3rd rings $=0$, 2, 4; and so on; for the $2n^{ic}$ the number of ovals in the 1st, 2nd, 3rd, ... $n-2^{th}$ rings $=0, 2, 4, 6, \ldots. 2n-10, 2n-8, 2n-6$. But in the consideration of the number of internal or external ovals, that nested oval which forms the inner boundary of a ring itself, belongs to the group of ovals in that region; hence the foregoing scheme becomes

$$1,\ 3,\ 5,\ 7, \ldots\ldots\ 2n-9,\ 2n-7,\ 2n-5,$$

and these groups are alternately internal and external ovals, or vice versa. Therefore for $n-2$ even (Fig. 5, Plate II, $n=4$),

$$\text{the number of internal ovals} = 2n-5+2n-9+\ldots+7+3 = \tfrac{1}{2}(n-1)(n-2);$$

for $n-2$ odd,

$$\text{the number of internal ovals} = 2n-5+2n-9+\ldots+5+1 = \tfrac{1}{2}(n-1)(n-2).$$

If in the generation of the $2n^{ic}$ one of the non-nested ovals lying outside the ellipse be taken at each stage as the generating oval, that is, if the *curve be derived by the 2nd mode of generation throughout*, there is obtained a similar arrangement of circuits in the annular regions. Beginning with 3, however, the series is reversed ($1, 2n-5, 2n-7, 2n-9, \ldots. 7, 5, 3$), since by this process after the generation of the 6^{ic}, the nest is built up from the outside inward; and the innermost ring contains not only one oval in accordance with the scheme, but also all, save one, which by the other process lay in that part of the plane exterior to the nest. Therefore for $n-2$ *even* when within the innermost ring,

the expression, C_{2n}, is positive (Fig. 6, Plate II), the number of internal ovals $= \frac{1}{2}(n-1)(n-2)$, but for $n-2$ *odd* the number of internal ovals

$$= n^2 + \tfrac{1}{2}(n-1)(n-2) - 1 > \tfrac{1}{2}(n-1)(n-2).$$

Combinations of the 1st and 2nd modes of generations give other types of curves. At each stage of the development of the curve, C_{2n}, from the conic, C_2, either mode of derivation may be adopted; hence from the quartic of which there is only one type, are derived two 6^{ics}, from each 6^{ic}, two 8^{ics}, and so on; the number increasing in geometrical ratio as n increases. Therefore for the $2n^{ic}$, if the two regular types just discussed be included, the number of types of curves with the maximum number of circuits and the maximum number of nested ovals, is 2^{n-2}. By a combination of the two modes, the nest is built up alternately from the inside outward and from the outside inward. If the curve, C_{2m-2}, is derived throughout the process by the first mode of generation, all of its nested ovals lie inside the ellipse, E_2. For the generation of the curve, C_{2m}, from this, let a change be made to the 2nd mode; then the *new* nested oval of C_{2m} lies outside the ellipse, and the annular region which corresponds to the curve, C_{2m}, is the one in which the ellipse is situated. As curves of higher order are generated all the *new* ovals appear in this region, and from it are cut off successively the *new* annular regions lying inside or outside the ellipse according as they are formed by the 1st or 2nd mode of generation. The curve is built up in such a manner that there are formed sets of annular regions, (1) a inside the ellipse, (2) a' outside the ellipse, (3) b inside the ellipse, (4) b' outside the ellipse, and so on. Except in those rings which are formed at stages where a change in the process occurs, the number of ovals in each annular region is the same as that in the corresponding region when one type of generation is used throughout,—that is in the last annular region formed whether outside or inside the ellipse there are $2n-6$ ovals. But consider the ring formed in the derivation of C_{2m} from C_{2m-2} where a change is made from the 1st to the 2nd mode of generation. The outer nested oval of C_{2m-2} is one of the $2m-4$ ovals arising from segments of g_{2m-4} and E_2, and lying inside the ellipse; and the generating oval of C_{2m-2} is one of the $2m-4$ ovals outside the ellipse. Thus $2m-5$ ovals are left in the region between the ellipse and the outer nested oval of C_{2m-2}. Hence as C_{2m} is derived from this $2m-2^{ic}$, an annular region is formed which contains $2m-5+2m-2$ ovals inside the ellipse and $2m-3$ lying outside. If the second process is continued through

the generation of C_{2p-2}, the number of ovals in each new annular region formed follows the regular scheme up to this stage, and the appearance of every group of $2q-6$ ovals in a ring outside the ellipse ($2q = 2m+2,\ 2m+4, \ldots .\ 2p-2$), is accompanied by the appearance of a group of $2q-4$ ovals in the region containing E_2 and inside the ellipse. Hence this ring contains

$$2m-5+\sum\limits_{m+1}^{p-1}{}_{q}\ 2q-4+2p-4 \text{ ovals inside the ellipse and } 2p-5 \text{ outside.}$$

If at this stage a change is made back to the first type of generation, one of the $2p-4$ ovals inside the ellipse must be taken as the generating oval, g_{2p-2}, and hence in the new annular region formed for C_{2p}, a region which lies inside the ellipse, there are $2m-5+\sum\limits_{m+1}^{p-1}{}_{q}\ 2q-4+2p-5$ ovals, and in the region which contains the ellipse $2p-3$ ovals inside the ellipse and $2p-2+2p-5$ outside. Hence as C_{2n} is derived, if there is a change from the 1st mode of generation to the 2nd in the derivation of C_{2m}, from the 2nd to the 1st in the derivation of C_{2p}, from the 1st to the 2nd in the derivation of C_{2s}, and so on, the scheme of arrangement of the ovals in the annular regions beginning with the innermost ring is represented by the following sets of groups, $a, a', b, b', c, c', \ldots .,$

$$(a)\,|\,\underrightarrow{0,\ 2,\ 4,\ 6,\ 8, \ldots .\ 2m-8}\,|,\ (b)\,|\,(2m-5+\sum\limits_{m+1}^{p-1}{}_{q}\ \underrightarrow{2q-4+2p-5),}$$
$$2p-4,\ 2p-2, \ldots .\ 2s-8\,|,$$

$$(c)\,|\,(2s-5+\sum\limits_{s+1}^{t-1}{}_{q}\ 2q-4+2t-5),\ 2t-4,\ 2t-2, \ldots .\ 2n-6,$$
$$(2n-2+2n-3+\sum\limits_{t+1}^{n}{}_{q}\ 2q-4+2t-5)\,|,$$

$$(b')\,|\,2t-8, \ldots .\ \underleftarrow{2s-2,\ 2s-4,\ (2s-5+}\sum\limits_{p+1}^{s-1}{}_{q}\ 2q-4+2p-5)\,|,$$

$$(a')\,|\,\underleftarrow{2p-8, \ldots .\ 2m-2},\ 2m-4\,|;$$

or if in the enumeration of the ovals in the annular regions the nested oval

which forms the inner boundary of the ring be included, the foregoing scheme becomes

$$(a) \,|\, 1,\ 3,\ 5,\ 7, \ldots . 2m - 7 \,|,\ (b) \,|\, (2m - 5 + \sum_{m+1}^{p-1} {}_q\, 2q - 4 + 2p - 4),$$
$$2p - 3,\ 2p - 1, \ldots . 2s - 7 \,|,$$

$$(c) \,|\, (2s - 5 + \sum_{s+1}^{t-1} {}_q\, 2q - 4 + 2t - 4),\ 2t - 3,\ 2t - 1, \ldots . 2n - 5,$$
$$(2n - 1 + 2n - 3 + \sum_{t+1}^{n} {}_q\, 2q - 4 + 2t - 5) \,|,$$

$$(b') \,|\, 2t - 7, \ldots . 2s - 1,\ 2s - 3,\ (2s - 4 + \sum_{p+1}^{s-1} {}_q\, 2q - 4 + 2p - 5) \,|,$$
$$(a') \,|\, 2p - 7, \ldots . 2m - 1,\ 2m - 3 \,|.$$

If the 2nd mode of generation is applied first, then the region which includes the ellipse corresponds to the 6^{ic}. The foregoing scheme applies, but $2m$ has the value 6, and group (a) disappears.

If the annular regions be considered in the order of their formation but that region in which the ellipse is situated be left to the last, the series of numbers representing the groups of ovals is $(a) \,|\, 1, 3, 5, 7, \ldots . 2m - 7 \,|$, $(a') \,|\, 2m - 3,\ 2m - 1 \ldots . 2p - 7 \,|$, etc. Comparison with the regular series obtained when the first mode of generation is employed throughout shows that after $2m - 7$, each number in this series is greater by 2 or by a quantity exceeding 2, than the corresponding number in the regular series. If in the last region of each set, $a, a', b, b', \ldots .$ the expression, C_{2n}, is negative, or more briefly if the last region in each set is negative except in that set in which the ellipse is situated, the series which represents the number of *internal* ovals is

$$\underset{\text{or } 3,\, 7,\, 11,}{1,\ 5,\ 9,} \ldots . 2m - 11,\ 2m - 7,\ 2m - 3,\ 2m + 1, \ldots . 2p - 7,\ 2p - 3, \ldots \ldots$$
$$2s - 7,\ 2s - 3, \ldots \ldots$$

in exact agreement with the regular series representing the number of internal ovals when the 1st mode of generation is used throughout. For though the first region of (a') is negative the number of its ovals exceeds by 4 the number in the last region of (a), and as the first regions of all the other sets $b, b', c, c', \ldots .$ are necessarily positive there can be no break in the series as passage is made from set to set. But when there is a change made from one mode of generation to the

other, two regions which are formed consecutively are separated by other regions as the process is continued, and hence the expression C_{2n} may be negative in both these regions. This apparently introduces the possibility of the existence of a series in which some of the numbers are less than the corresponding numbers in the regular series, and in this case the number of internal ovals might be less than $\frac{1}{2}(n-1)(n-2)$. For example, if the last region of (b') and the first region of (c) are both negative, the series will contain a sequence of the form, $2t-7$, $(2t-4+\,..)$ $2t-1$, $2t+3,\ldots$, (A), in which the numbers beginning with $2t-1$ are less by 2 than the corresponding numbers in the sequence $2t-7$, $2t-3$, $2t+1$, $2t+5\ldots\ldots$ But the conditions for the existence of such a sequence as (A) demand the existence of a preceding sequence of the form $2l-9$, $2l-3$, $2l+1$, $2l+5$ instead of the regular sequence $2l-9$, $2l-5$, $2l-1$, $2l+3$. Thus the numbers in the series which precede such a sequence as (A) exceed by 2 the corresponding numbers in the regular series, and hence the numbers following the sequence will be the same as the corresponding numbers in the regular series.

Suppose such a sequence as $2p-7$, $(2p-4+\,..)$, $2p-1$, $2p+3$ occurs in the passage from the set (a') to the set (b). The last region of (a') and the first region of (b) are both negative. Since the first region of (b) is negative, the last region of (a) must be positive. The last negative region of (a) then contains $2m-9$ ovals, and hence in the series of numbers, representing the groups of internal ovals there is a spring from $2m-9$ to $2m-3$, preceding the succession $2p-7$, $(2p-4+\,..)$, $2p-1$, $2p+3$.

$(a)\ldots\ldots\ldots+(b)-\ldots\ldots\ldots-(c)\ldots\ldots\ldots+(d)\ldots\ldots\ldots(e)\ldots\ldots\ldots$

$(d')\ldots\ldots\ldots(c')+\ldots\ldots\ldots+(b')-\ldots\ldots\ldots(a')-\ldots\ldots\ldots-$

No such break can occur again in the series so long as the last region in each set considered is negative, for as a prerequisite to such a sequence as the above, the first region of some set must be negative and this is impossible unless the last region in the preceding contiguous set be positive. Let the first set whose last region is positive be (c). The last region of (b') is negative, hence the first region of (c') is positive. Therefore in the passage from (c) to (c') there is a jump over two annular regions, that is, from a group of $2l-9$ internal ovals to a group of $2l-3$. If the last region of (c') is positive, the series representing the groups of internal ovals goes on regularly except that in the 1st region of

(d), which is negative since the last region of (c) is positive, the number of internal ovals is greater than it would have been had there been no change in the mode of generation. This regularity continues until the last region in some set is negative, and at this stage occurs again a sequence of the form $2r - 7$, $(2r - 4 + \ldots)$, $2r - 1$, $2r + 3 \ldots.$ In every case a sequence of the above form is preceded by a succession of the form $2l - 9$, $2l - 3$, $2l + 1$, $2l + 5$. Thus in the series representing the groups of internal ovals there may be numbers exceeding but none less than the corresponding numbers in the regular series. Hence since the series of numbers which represent the groups of internal ovals on curves derived by both modes of generation is either greater than or equal to the series obtained when the 1st mode of generation is used throughout, the number of internal ovals $\overline{\gtrless}\ \frac{1}{2}(n - 1)(n - 2)$.

The *third mode of generation* in which the new nested oval is taken as the generating oval, is not applicable to the generation of curves of degree lower than 8, for the 6^{ic} is the first curve whose generating oval can be a nested oval. Hence the first application of the process must be preceded by the use of the first or second mode of generation. Each application of the process reduces the number of nested ovals by 2. For suppose the new nested oval of C_{2m-2} cuts the ellipse; then on C_{2m} there is no *nested* oval representing this, and moreover none of the $4m - 4$ ovals arising from the segments of the ellipse and this generating oval encloses either the ellipse or the nested ovals lying inside the ellipse; that is, the nested oval contributed to the nest by C_{2m-2} disappears as such, and no new nested oval is added by C_{2m}. It is evident that since there is no nested oval among the new ovals formed at this stage, the third mode of generation cannot be applied twice in succession. Thus the different types of curves which are derived in part by the 3rd mode of generation are those obtained by the 1st and 3rd modes of generation or by the 2nd and 3rd, or by combinations of all three modes, or by combinations of these with the fourth. If the curves be derived by the 1st and 3rd *modes of generation*, let the 3rd mode be introduced for the first time for the generation of the curve C_{2m}. The new nested oval of C_{2m-2} lies inside the ellipse, and in the new annular region formed there are $2m - 8$ ovals, and in the region between the nested ovals and the ellipse there are $2m - 5$ ovals. The new nested oval of C_{2m-2} is to be taken as the generating oval. Hence in one of the $4m - 4$ regions bound by segments of E_2 and g_{2m-2} there are $2m - 5$ ovals, and therefore of the $4m - 4$ new ovals appearing on C_{2m}, one contains $2m - 5$ others, and this lies inside the ellipse. The

remaining $4m-5$ are distributed with respect to the ellipse just as the corresponding $4m-5$ ovals would have been distributed had the first mode of generation been used instead of the third; so the process can be continued as if the third type had not been introduced except that a nested oval cannot be taken as the generating oval. Hence the general arrangement of the ovals of the curve, C_{2m}, which is derived throughout by the 1st method of generation is affected by the introduction of the third method for the generation of C_{2m} only in the following manner. *The ring which by the continuation of the first process would have contained $2m-6$ ovals disappears, and in place of the two nested ovals bounding it,*

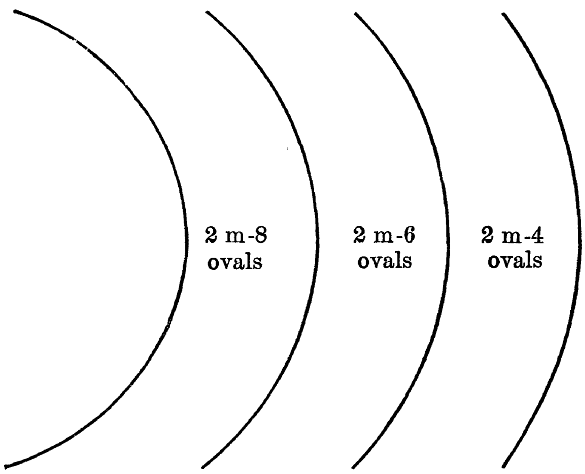


Fig. 5.

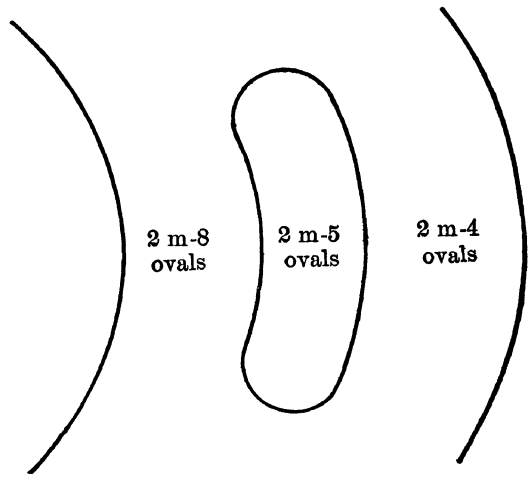


Fig. 6.

with the $2m-6$ ovals lying between them, there is now one oval enclosing $2m-5$ others. Thus three annular regions are thrown into one, and the arrangement of circuits represented by Fig. 5 becomes that indicated by Fig. 6. Or if the process stops here, the two outer rings become one with the region outside the nest.

If the annular region which is composed of the three is positive, it contributes the same number of internal ovals as the middle region would have contributed, if the three were distinct, namely $2m-5$. If negative, it yields the same number that would have been given by the other two regions. In the enumeration of the internal ovals the nested oval which forms the inner boundary of a negative ring must be included, and though the two regions in question would yield two such internal ovals and the composite region only one,

yet as compensation for the other there is the oval which contains the $2m - 5$ other ovals. If the new nested oval of C_{2m+2} be taken as the generating oval for the derivation of C_{2m+4}, that is, the 3rd mode of generation applied again, then there are combined into one the five annular regions which would have appeared had the first mode of generation been used throughout; but in this composite region there are besides $2m - 8 + 2m - 4 + 2m$ ovals, one oval enclosing $2m - 5$ others, and another oval encircling $2m - 1$ ovals. The arrangement is indicated by Fig. 7. It is evident that the number of internal ovals is not altered by the introduction of the third mode of generation, and hence every

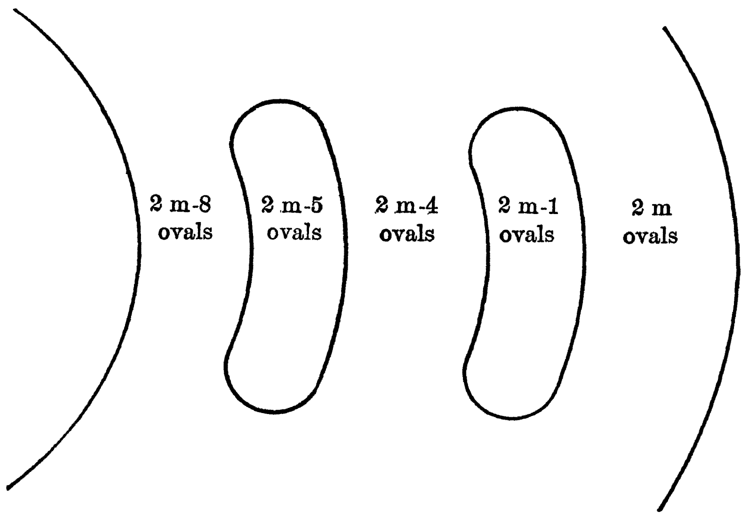


FIG. 7.

combination of the 1st and 3rd modes of derivation gives a curve with $\frac{1}{2}(n - 1)(n - 2)$ internal ovals.

The arrangement of the circuits of a curve derived by the 2nd mode of generation or by a combination of the 1st and 2nd modes of generation is modified by the introduction of the 3rd mode in a manner similar to that in the preceding case. Suppose the 3rd mode is introduced for the derivation of C_{2m}. Either the 1st or 2nd process must have been used for the derivation of C_{2m-2}. If this process had been continued for C_{2m} also, an annular region would have been formed containing $2m - 6$ ovals between the two nested ovals. The use of the 3rd method causes the disappearance of these two *nested* ovals as such and introduced in the place of them, with the $2m - 6$ ovals between them, one oval enclosing $2m - 5$ others. And this is the only alteration produced. The

conclusion is easily deduced as in the preceding case that the number of internal ovals on curves derived by the 2nd and 3rd modes of generation or by all three modes combined, is not less than $\frac{1}{2}(n-1)(n-2)$.

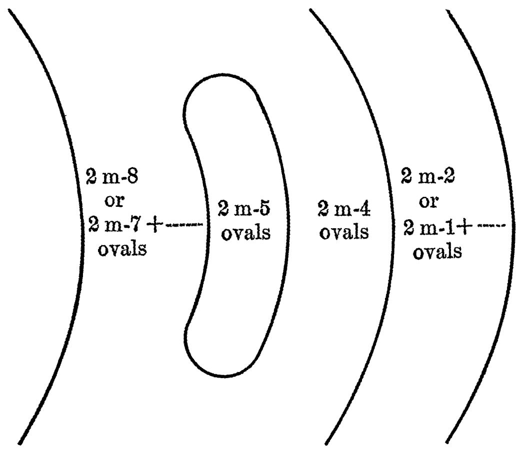


FIG. 8.

Although the 3rd mode of generation cannot be applied twice in succession, it need not be followed by the first or second, for the ellipse may be cut by the oval which contains $2m-5$ others. This introduces a *fourth mode of derivation,*

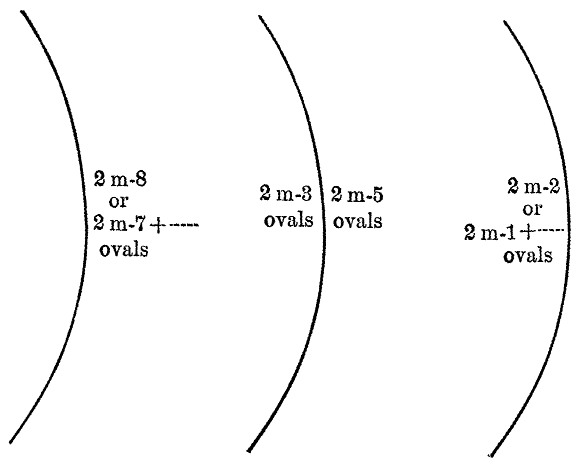


FIG. 9.

closely related however to the 3rd type, for though the generating oval is not one of the nested ovals, yet it does include others. It is not difficult to see that if the arrangement of the ovals in the annular regions formed at the stages

C_{2m-2}, C_{2m}, C_{2m+2}, C_{2m+4} on a curve derived by the 1st, 2nd and 3rd modes be represented by Fig. 8, the use of the 4th mode for the derivation of C_{2m+2} will modify the arrangement to that indicated by Fig. 9. That is, there is a combination in pairs of the four annular regions which would have appeared if instead of the 3rd and 4th modes of generation the one preceding the 3rd had been used; the first and third are united, and the second and fourth. If one region is negative, the other is positive, so that the same number of internal ovals is obtained as if the ovals were distributed in the four regions. Hence in this case, the introduction of the 4th mode of generation does not decrease the number of

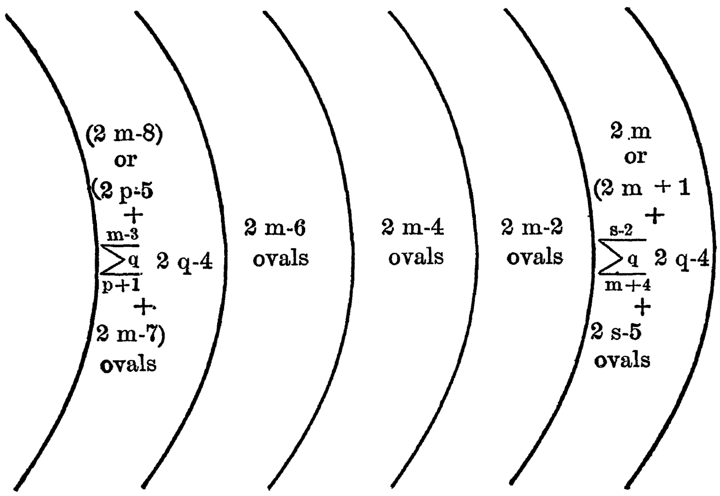


FIG. 10.

external ovals. If the 4th mode of generation be followed by the 3rd instead of the 1st or 2nd, then a somewhat different arrangement is produced. There are combined into one the five annular regions which would have appeared, if the mode by which C_{2m-2} was derived had been continued for the generation of C_{2m}, C_{2m+2}, C_{2m+4}, C_{2m+6}. Instead of the arrangement represented by Fig. 10, the result would be that indicated by Fig. 11.

If all possible combinations be made of the four modes of generation, various arrangements of the circuits are obtained, but the investigation of the preceding combinations makes it evident that the number of internal ovals on the derived curve is equal to the number on a curve which is generated by the 1st and 2nd modes only and hence is not less than $\frac{1}{2}(n-1)(n-2)$.

The appearance of a curve derived by the 1st or 2nd mode of generation, or by a combination of the two, is only slightly modified by an occasional intro-

duction of the 3rd mode of derivation; but an extensive use of the 3rd method gives a curve differing greatly in form from that derived by the 1st or 2nd mode alone, or by the 1st and 2nd together. For example, if for the derivation of the 8^{ic} from the 6^{ic} which is generated by the 1st mode of generation, the 3rd mode be employed, and for the derivation of curves of higher order the 4th and 3rd modes be used alternately, then no nest whatever is built up. The type of curves obtained is the same as that derived by the exclusive use of the 1st mode of generation in the Harnack process. The 8^{ic} has one oval containing three, and the remaining ovals are external to one another; the 10^{ic} has one oval

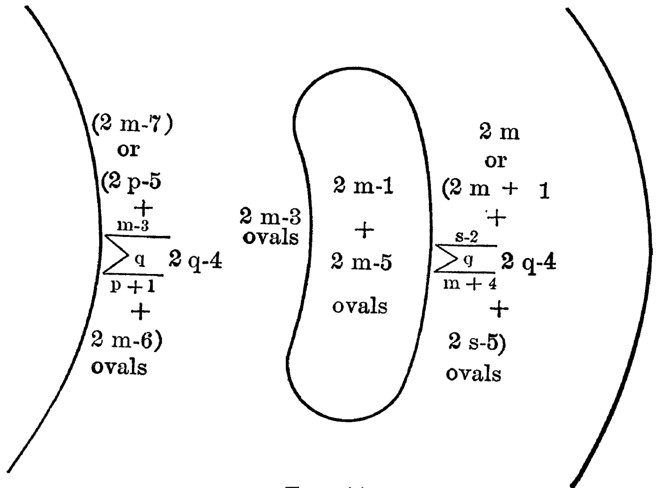


Fig. 11.

containing 6 and the remainder external to one another, and so on. If the 1st mode of generation is combined with the 3rd and 4th, but after the generation of the 8^{ic} not applied twice in succession, there are derived in this way $m-2$ other types of curves of order $4m$, and $l-2$ other types of curves of order $2(2l-1)$, which agree with types obtained by the Harnack process. But no other types of the Harnack curves are derived by the use of the 2nd mode instead of the 1st or by the use of both the 1st and 2nd.

It is evident from the method of generation that the number of internal ovals can be diminished in the same manner as in the generation of curves by the Harnack process, and also that this diminution can in no case be accompanied by an increase in the number of external ovals. Therefore the number of external ovals is not greater than $n^2+\frac{1}{2}(n-1)(n-2)$ even if the number of circuits falls short of the maximum number.

Curves Derived by the Modified Forms of the Hilbert Method.—The use of the *hyperbola* or *parabola* as auxiliary curve, since these can be projected into the ellipse, can certainly yield no type of $2n^{ic}$ differing from those obtained by the Hilbert process. Neither does the *degenerate conic.* This, however, requires a special proof. But when the pair of straight lines is substituted for the ellipse as the auxiliary curve, passage can be made at any time to the Harnack process, by the mere disregard of one of the lines, and a return to the original mode of derivation can be effected after the generation of any curve of even order. Thus new types of curves may arise by a combination of the two processes, but it can certainly be shown that no type of $2n^{ic}$ obtained has less than $\frac{1}{2}(n-1)(n-2)$ internal ovals.

If it could be shown that all non-singular curves with the maximum number of circuits can be generated by this method of small variation, the proof of the validity of the law for these remaining cases would establish it for all non-singular curves of even order. But as yet there is no formal proof that the list of such curves is exhausted by the types considered.

CONCLUSION.

There are several other forms in which the theorem can be stated that are of interest, either as facts resulting from the theorem if established in its preceding form, or as statements which may afford a better starting point for the proof of the theorem. A few of these equivalent forms are obtained by a consideration of the *Theory of the Characteristic,* which though apparently yielding no results toward the proof of the theorem, bears a most interesting relation to the problem. The theory as given by Kronecker* is purely algebraic; he proved that for any system of algebraic functions satisfying certain conditions, there exists a number derived algebraically which is invariant for that system. Dyck,† however, was led by a study of Kronecker's investigations to a *geometrical* definition of a characteristic number associated with a manifold, a number which is built up as the manifold itself is developed. He showed that if the manifold can be expressed algebraically, it can be developed by processes which also are capable of algebraic expression. The whole geometrical configuration thus

* *Ueber Systeme von Functionen mehrerer Variabeln,* Werke I, 1869, pp. 175–212, 213–226. *Ueber die Characteristik von Functionen Systemen,* Werke II, 1878, pp. 71–82.

† *Beiträge zur Analysis Situs,* I, II, III, Berichte der K. Sächs. Gesellschaft der Wissenschaften (Math. Phys. Classe) 1885, 1886, 1887. Math. Ann., vol. 32.

introduces a system of algebraic functions subject to certain conditious. Dyck proved that the characteristic number associated with the manifold can be derived from this system of functions and that the number is identical with the Kronecker characteristic of the system.

In Dyck's Theory the manifold is regarded as made up of elements, and to each element is assigned the characteristic $+1$. Whatever be the process of generation of the manifold,

(*a*) the appearance of a new element contributes $+1$ to the characteristic;

(*b*) the vanishing of an element contributes -1;

(*c*) the separation of an element into two pieces contributes $+1$;

(*d*) the joining of two elements, or of two parts of the same element, contributes -1.

For a one-dimensional manifold,—that is, a figure composed of lines,—the element is a broken piece of a curve. For a two-dimensional manifold, the element may be a part of a plane or of a surface that can be developed as a plane. The manifold suggested by any problem relating to the arrangement of the circuits on a plane curve, $f(xy) = 0$, of even order, with no singularities and with the maximum number of circuits, is obviously the two-dimensional manifold determined as the parts of the plane lying inside the curve,—that is, the parts of the plane where $f < 0$. The element is a piece of the plane bounded by a non-singular closed circuit (Fig. 12), and to this is assigned the characteristic $+1$. If the figure is initially non-existent, its characteristic is zero, and as the manifold is generated the characteristic is increased by unity as an element appears or separates into two, and is diminished by unity as an element vanishes, or as two elements or two parts of the same element unite. Therefore the characteristic of the manifold is equal to the sum of the characteristics of the separate parts which make up the manifold, and is independent of the mode of generation. Thus in the given manifold a part of the plane bounded by a single oval has the characteristic $+1$, whether it arises from a single element in the process of generation or from the union of several elements. Hence each external oval on the curve contributes to the manifold a region whose characteristic is $+1$. If two parts of such a piece of the plane unite and thus enclose in the midst of a region in which f is negative, a region where f is positive, their union is

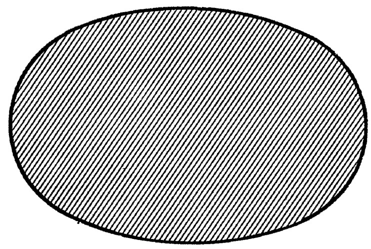


FIG. 12.

marked by -1 in the characteristic. But this is just the way in which an external oval makes its appearance. Hence the presence of each internal oval on the curve diminishes the characteristic of that piece of the plane to which it belongs by unity. The whole number of circuits on the $2n^{ic}$ is $n^2 + \frac{1}{2}(n-1)(n-2)$, and hence according to the theorem which ascribes the minimum limit $\frac{1}{2}(n-1)(n-2)$ to the number of internal ovals, the characteristic cannot exceed $n^2 + \frac{1}{2}(n-1)(n-2) - \frac{1}{2}(n-1)(n-2)$ or n^2, if the maximum number of circuits is present. It has been noted that a decrease in the number of internal ovals cannot be accompanied by an increase in the number of external ovals. Hence the characteristic cannot be greater than $n^2 + \frac{1}{2}(n-1)(n-2)$ even if the number of branches is not the maximum number.

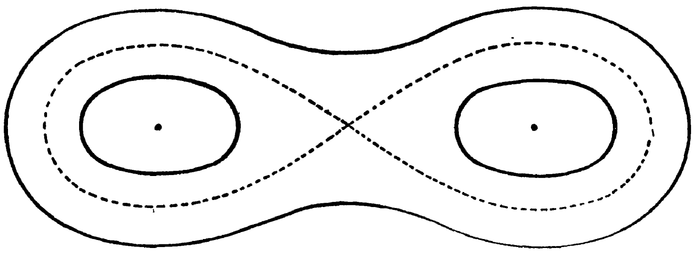


FIG. 13.

The process of generation of such a manifold can be expressed analytically by considering the manifold as one of a singly infinite system of manifolds. Let the curve $f = 0$ be obtained as one of the pencil $f = \lambda$. Then the region where f is negative decreases as λ decreases, and for some value of λ sufficiently near $-\infty$, the curve disappears altogether. The characteristic of the corresponding manifold is zero. Let λ increase from this value. The curve makes its appearance as an isolated point spreading into a circuit as λ continues to increase. Other circuits also may come into existence in the same way. Thus an isolated point gives rise to a part of the plane and hence has the value $+1$ for the characteristic. Two circuits may unite, producing a node (Fig. 13) and thus join two pieces of the plane together, or a single circuit may cut itself (Fig. 14), and so unite parts of the same region of the plane. In either case the node is marked by -1 in the characteristic. As λ increases, an internal oval shrinks until it becomes an isolated point and then passes out of existence, or it may cut itself in such a way that at the next stage it breaks into two internal ovals. In all cases the isolated points which present themselves in the region $f < 0$ contribute $+1$ to the characteristic, and the nodes -1. The singular points of the system

are given by $f_1 = 0, f_2 = 0$, and are nodes with real tangents or isolated points according as $\begin{vmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{vmatrix}$ is negative or positive. If the notation $\left[\begin{vmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{vmatrix}\right]$ be used to represent $+1, 0, -1$ according as the determinant is positive, zero, or negative, the characteristic of the manifold can be written $X = \Sigma \left[\begin{vmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{vmatrix}\right]$, the summation taken over all points $f_1 = 0, f_2 = 0, f < 0$; and Dyck shows that this is the same as the Kronecker characteristic for the system of functions f, f_1, f_2. Since the characteristic cannot exceed n^2 if $f = 0$ be a curve (order $2n$) with the maximum number of circuits, it follows that the number of isolated points passed over in the system $f = \lambda$ from $\lambda = -\infty$ to $\lambda = 0$ exceeds the

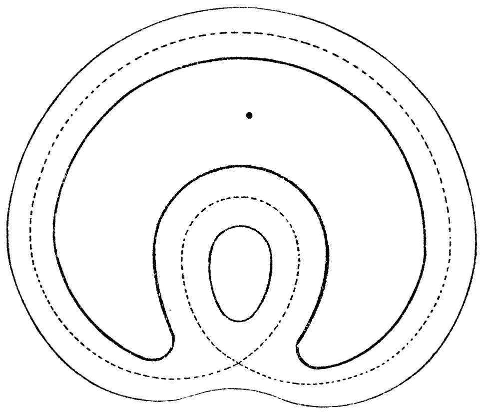


FIG. 14.

number of nodes by a quantity less than, or equal to n^2. And if $f = 0$ has not the maximum number of circuits, the excess of the number of isolated points over the number of nodes in the region $f < 0$ is either less than or equal to $n^2 + \frac{1}{2}(n-1)(n-2)$.

The relations between the critic centres of the pencil $f = \lambda$ thus obtained in applying the Theory of the Characteristic to an interpretation of the problem are interesting, but afford no clue to the solution. There is even some indication that the theory is not the most promising instrument of proof, for this is applicable to curves of even order only, and though the theorem on the minimum limit of the number of internal ovals is stated for these curves alone, it most probably can be extended to include curves of odd order also. It has been seen that on a non-singular $2n + 1^{ic}$ with the maximum number of branches the ovals may lie each outside the others; but even in this case they may satisfy

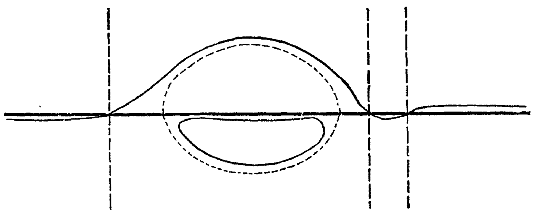
the definition of "internal ovals," for the odd circuit divides the plane into two regions both infinite, in one of which C_{2n+1} is positive, and in the other, negative. And as a matter of fact, no $2n+1^{ic}$ with fewer than $\frac{1}{2}(n-1)(n-2)$ interval ovals presents itself directly by either of the two processes of generation discussed.

There are still other forms in which the theorem can be stated, the most interesting of which is perhaps one relating to the number of regions into which the plane is divided by the curve. It has been shown that if the maximum number of branches is present, then the curve must have at least $\frac{1}{2}(n-1)(n-2)$ *internal* ovals, and whether the number of branches present is the maximum or less than the maximum, the curve can not have more than $n^2+\frac{1}{2}(n-1)(n-2)$ *external* ovals. By a similar line of reasoning it can be proved that whatever the number of circuits, the number of internal ovals can not exceed $n^2+\frac{1}{2}(n-1)(n-2)-1$; and hence if the maximum number of circuits is present the number of external ovals can not fall below $\frac{1}{2}(n-1)(n-2)+1$.

Therefore if the maximum number of branches is present,
the number of regions in which the expression C_{2n} is positive $\geqq \frac{1}{2}(n-1)(n-2)+1$,
and the number of regions in which C_{2n} is negative $\geqq \frac{1}{2}(n-1)(n-2)+1$;
and whatever the number of circuits,
the number of regions in which C_{2n} is positive $\leqq n^2+\frac{1}{2}(n-1)(n-2)$
and the number of regions in which C_{2n} is negative $\leqq n^2+\frac{1}{2}(n-1)(n-2)$.
From these statements, it seems that any limitation on the arrangement of the circuits is of a dual nature; and it is worthy of note that no modification of these statements is necessary, if the sign of C_{2n} be so chosen that it is negative at infinity.

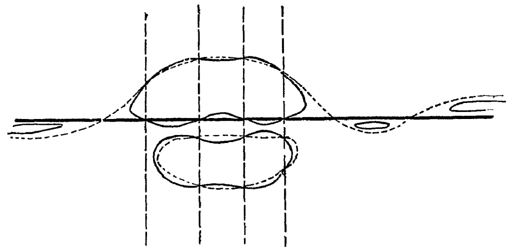
The statement of the theorem in this last form suggests that there may be some underlying relation to the theory of Multiply-connected Surfaces.

In the figures of Plates I and II the curves are much distorted, inflexions being inserted where none exist, in order to bring the figures within the scale of the paper. The figures represent the distribution of the ovals of the curves accurately only with respect to the *number* in different regions of the plane. In the drawing of the figures representing curves of the type, $C_{2n}\equiv C_{2n-2}\cdot E_2+\delta\prod_{i=1}^{i=2n} l_i=0$, the straight lines l_i are omitted.



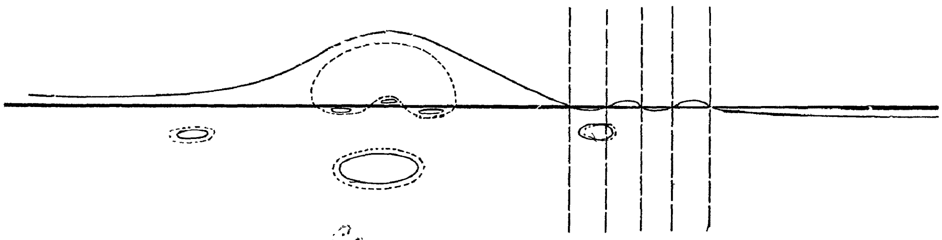
$$C_3 \equiv C_2 \, v + \delta \prod_{i=1}^{i=3} l_i = 0.$$

FIG. 1.



$$C_4 \equiv C_3 \, . \, v + \delta \prod_{i=1}^{i=4} l_i = 0.$$

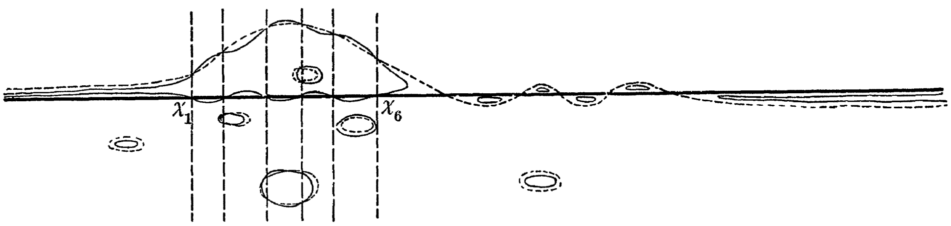
FIG. 2.



C_5

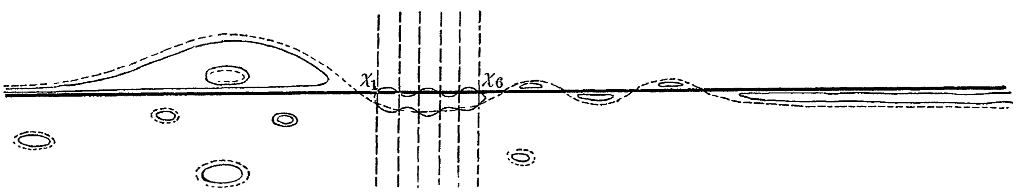
$$\cancel{C_6} \equiv C_4 \, . \, v + \delta \prod_{i=1}^{i=5} l_i = 0.$$

FIG. 3.



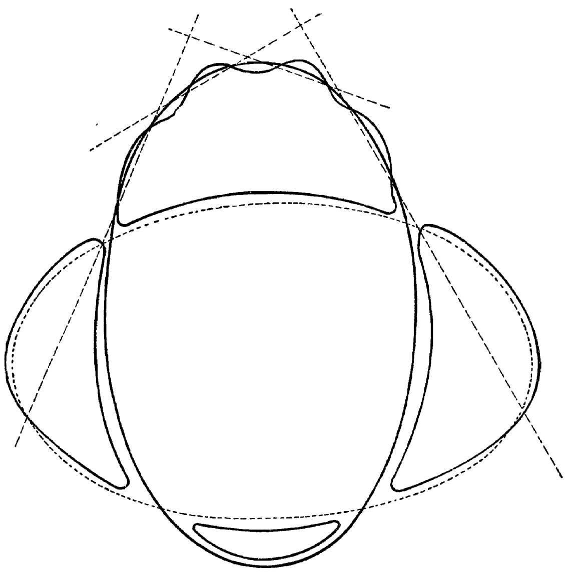
$$C_6 \equiv C_5 \, . \, v + \delta \prod_{i=1}^{i=5} l_i = 0.$$

FIG. 4.



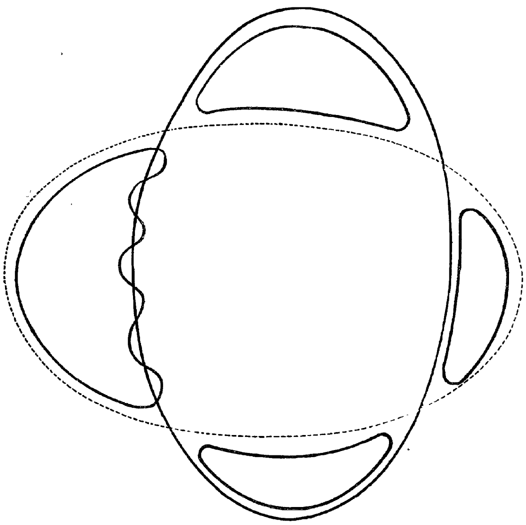
$$C_6 \equiv C_5 \, . \, v + \delta \prod_{i=1}^{i=6} l_i = 0.$$

FIG. 5.



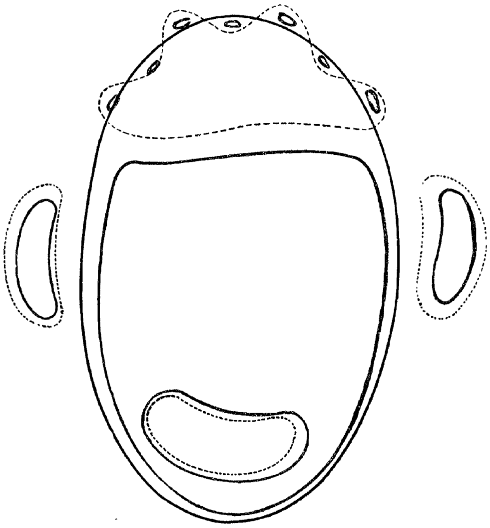
$\underline{C_4} \equiv \underset{\cdots}{C_2} \cdot \underline{E_2} + \delta \prod_{i=1}^{i=4} l_i = 0.$

FIG. 1.



$\underline{C_4} \equiv \underset{\cdots}{C_2} \cdot \underline{E_2} + \delta \prod_{i=1}^{i=4} l_i = 0.$

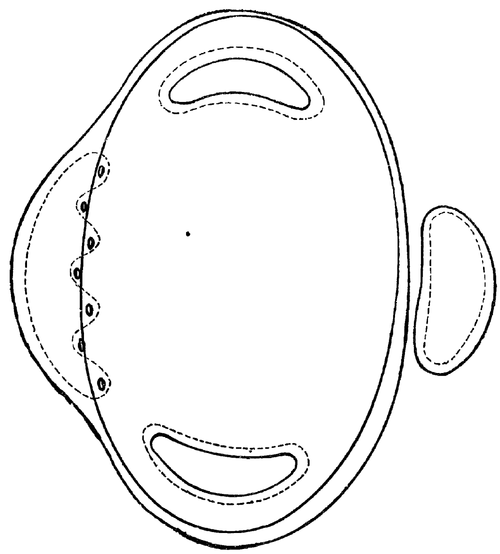
FIG. 2.



$\underline{C_6} \equiv \underset{\cdots}{C_4} \cdot \underline{E_2} + \delta \prod_{i=1}^{i=6} l_i = 0.$

C_6 from C_4 by 1st mode of generation.

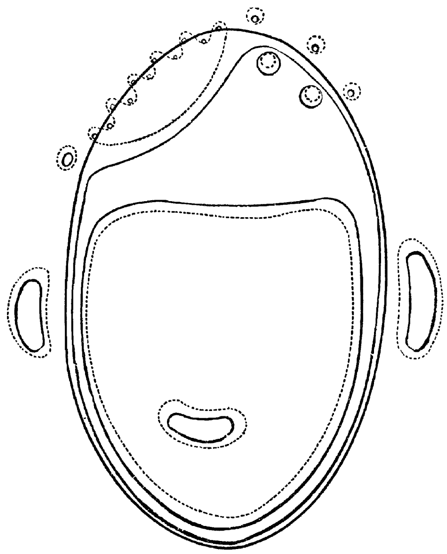
FIG. 3.



$\underline{C_6} \equiv \underset{\cdots}{C_4} \cdot \underline{E_2} + \delta \prod_{i=1}^{i=6} l_i = 0.$

C_6 from C_4 by 2nd mode of generation.

FIG. 4.

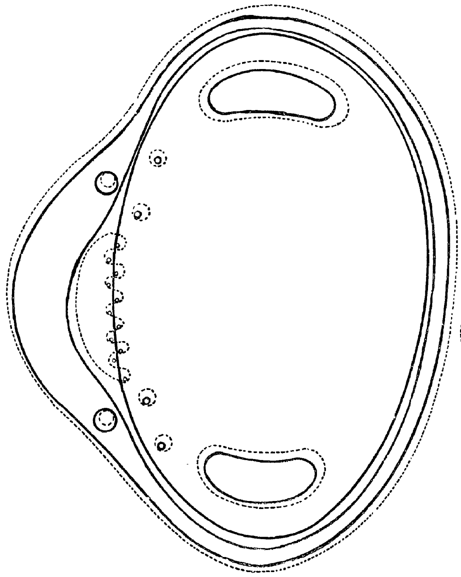


$\underline{C_8} \equiv \underset{\cdots}{C_6} \cdot \underline{E_2} + \delta \prod_{i=1}^{i=8} l_i = 0.$

C_8 from C_6,

C_6 from C_4 by 1st mode of generation.

FIG. 5.



$\underline{C_8} \equiv \underset{\cdots}{C_6} \cdot \underline{E_2} + \delta \prod_{i=1}^{i=8} l_i = 0.$

C_3 from C_6,

C_6 from C_4 by 2nd mode of generation.

FIG. 6.